A study of the magnetohydrodynamic boundary layer on a flat plate

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This paper is concerned with the boundary layer on a semi-infinite flat plate in a uniform stream of conducting fluid, with a magnetic field in the stream direction such that the Alfvén speed is less than the undisturbed fluid speed. Series solutions are derived which are applicable for large and small values of the electrical conductivity, and which give a guide as to the validity and limitations of theories which assume the fluid to have infinite or zero conductivity.

1. Introduction

The subject of this paper is the derivation of solutions of the equations governing a certain problem in magnetohydrodynamics. This problem, discussed in detail by Greenspan & Carrier (1959) in a paper hereafter referred to as **I**, concerns the steady two-dimensional flow of a viscous incompressible electrically conducting fluid of constant properties past a semi-infinite rigid plate. The applied magnetic field is uniform and in the direction of the undisturbed stream, which is parallel to the plate and perpendicular to its edge.

As shown in **I** and recapitulated briefly in the Appendix to this paper, the boundary-layer equations governing the flow are

$$f''' + ff'' - \beta g g'' = 0, \tag{1.1}$$

$$g'' + \epsilon(fg' - f'g) = 0, \tag{1.2}$$

with boundary conditions

$$f(0) = f'(0) = g(0) = 0, \quad f'(\infty) = g'(\infty) = 2. \tag{1.3}$$

Here $\epsilon = \sigma\mu\nu$ is the ratio of the viscous to the magnetic diffusivity and is proportional to the conductivity of the fluid, and $\beta = \mu H_0^2/\rho U_0^2$ is the square of the ratio of the Alfvén speed to the fluid speed in the undisturbed flow. The fluid has density ρ, kinematic viscosity ν, electrical conductivity σ and magnetic permeability μ, and in the undisturbed stream the speed is U_0 and the magnetic field intensity is H_0. The functions $f(\eta)$ and $g(\eta)$ describe the velocity field and the magnetic field respectively, where

$$\eta = \tfrac{1}{2}(U_0/\nu x)^{\frac{1}{2}} y \tag{1.4}$$

is the usual Blasius non-dimensional variable, x and y being distances measured along and perpendicular to the plate from its leading edge. The velocity components are

$$u = \tfrac{1}{2}U_0 f', \quad v = \tfrac{1}{2}(U_0 \nu/x)^{\frac{1}{2}}(f - \eta f'), \tag{1.5}$$

and the magnetic field components are

$$H_x = \tfrac{1}{2}H_0 g', \quad H_y = \tfrac{1}{2}H_0(\nu/U_0 x)^{\frac{1}{2}}(g - \eta g'). \tag{1.6}$$

The method used in I to tackle the problem was to replace the fundamental equations by Oseen equations, linearized equations obtained by assuming that the convective velocity and magnetic fields may be replaced by their free-stream values. For $\beta < 1$ the method led to plausible results, although they did not agree very satisfactorily with certain numerical solutions of equations (1.1) and (1.2). For $\beta > 1$ the whole formulation of the problem breaks down. The Alfvén speed is then greater than the fluid speed and disturbances penetrate upstream ahead of the plate, so it is no longer possible to describe the flow completely in terms of the Blasius variable based on the leading edge. This was confirmed in I by a study of the Oseen equations applicable to a flat plate of finite length. The solutions for $\beta < 1$ obtained in I, and an investigation of the nature of the equations when $1-\beta$ is small and $\epsilon = 1$, were interpreted by the authors as indicating that the entire flow becomes 'plugged' or brought to rest at the critical value $\beta = 1$.

Further discussions of the problem by means of the Oseen equations have been given by Carrier & Greenspan (1960) and by Greenspan (1960). The former paper treats unsteady flow conditions and the latter the flow past a finite plate with $\beta > 1$.

The scope of the present paper is limited to the range $\beta < 1$, with $1-\beta$ not small. Two solutions in series of the full boundary-layer equations (1.1) and (1.2) are obtained, valid for large and for small values of ϵ respectively. Several terms are calculated explicitly, each depending in a simple manner on the parameter β. The physical reason why such series expansions are possible is that when ϵ is large the magnetic boundary layer is effectively much thinner than the velocity boundary layer and when ϵ is small it is much thicker. The situation has many points of similarity to the study of heat transfer in a laminar boundary layer of non-conducting fluid, as discussed by Morgan & Warner (1956) and by Morgan, Pipkin & Warner (1958). The temperature distribution function and the Prandtl number there take analogous places in the equations to the magnetic field function $g(\eta)$ and the conductivity parameter ϵ.

Our series solutions appear to give reliable numerical values for the ranges $\epsilon > 10$ and $\epsilon < 0{\cdot}001$. Most practical applications will probably be covered by one or other of these cases. If results are required for some specific values of ϵ outside these ranges they may be obtained without undue labour by use of an electronic computer, but the presence of two arbitrary parameters ϵ and β means that anything like a complete coverage is a formidable task. The equations have simple solutions for $\epsilon = \infty$ and for $\epsilon = 0$, and perhaps the chief point of interest is to see how these limiting values are approached, since in many problems in magnetohydrodynamics it is most helpful to be able to assume that the electrical conductivity is either infinite or zero.

2. Equations for large conductivity

When the electrical conductivity is large it is convenient to make a slight change of variables, in order that the functions to be calculated shall as far as possible be independent of the parameter β. We write

$$\eta = (1-\beta)^{-\frac{1}{2}}\theta, \quad f(\eta) = (1-\beta)^{-\frac{1}{2}}p(\theta), \quad g(\eta) = (1-\beta)^{-\frac{1}{2}}q(\theta), \tag{2.1}$$

and

$$\epsilon = (1-\beta)\lambda. \tag{2.2}$$

As stated above, $1-\beta$ is supposed to be positive and not small. Then

$$f'(\eta) = p'(\theta), \quad f''(\eta) = (1-\beta)^{\frac{1}{2}}p''(\theta), \quad \text{etc.,}$$

and the equations (1.1) and (1.2) become

$$(1-\beta)p''' + pp'' - \beta qq'' = 0, \tag{2.3}$$

$$q'' + \lambda(pq' - p'q) = 0, \tag{2.4}$$

with boundary conditions

$$p(0) = p'(0) = q(0) = 0, \quad p'(\infty) = q'(\infty) = 2. \tag{2.5}$$

For a perfectly conducting fluid λ, like ϵ, is not only large but infinite. Equation (2.4) reduces to $pq' = p'q$ and the only solution satisfying (2.5) is

$$p(\theta) = q(\theta). \tag{2.6}$$

Equation (2.3) now becomes

$$p''' + pp'' = 0, \tag{2.7}$$

of which the required solution is

$$p(\theta) = B(\theta), \tag{2.8}$$

where $B(\theta)$ is the well-known Blasius function which governs the boundary layer on a flat plate in a non-conducting fluid.

If λ is finite this solution is incorrect in the vicinity of the plate. It implies that $q''(0) = B''(0)$, which is non-zero, while the boundary conditions (2.5) and equation (2.4) show that $q''(0) = 0$. It is clear that however large λ may be there is an inner part of the boundary layer in which the first term of (2.4) cannot be ignored. According to (2.8), when θ is small $p = O(\theta^2)$, $p' = O(\theta)$, $p'' = O(1)$ and similarly for $q(\theta)$, and so the terms of (2.4) become of comparable magnitudes when $\theta = O(\lambda^{-\frac{1}{3}})$. We must introduce an appropriately stretched co-ordinate to describe the inner layer. Since the transformed equation (2.4) must no longer contain λ explicitly we write

$$\theta = \lambda^{-\frac{1}{3}}\xi, \quad p(\theta) = \lambda^{-\frac{2}{3}}P(\xi), \quad q(\theta) = \lambda^{-\frac{2}{3}}Q(\xi). \tag{2.9}$$

It follows that $p'(\theta) = \lambda^{-\frac{1}{3}}P'(\xi)$, $p''(\theta) = P''(\xi)$, etc., and (2.3) and (2.4) become

$$\lambda(1-\beta)P''' + PP'' - \beta QQ'' = 0, \tag{2.10}$$

$$Q'' + PQ' - P'Q = 0, \tag{2.11}$$

with boundary conditions $P(0) = P'(0) = Q(0) = 0$. The boundary condition at $\theta = \infty$ is of no importance to the inner layer.

We may now seek solutions for p, q, P and Q in the form of series in descending powers of λ, and hope to be able to determine the arbitrary constants which arise in the integrations by matching the forms of the inner solutions for large ξ to the outer solutions for small θ. The form of the transformation (2.9) shows that it will be most convenient to carry out this matching by balancing corresponding second derivatives, since the relations between these do not involve λ explicitly.

3. Solutions for large conductivity

The first approximations $p_0(\theta)$, $q_0(\theta)$, $P_0(\xi)$, $Q_0(\xi)$ to the solutions are found by taking λ to be infinite in (2.3), (2.4), (2.10) and (2.11). As determined in (2.6) and (2.8)

$$p_0(\theta) = q_0(\theta) = B(\theta). \tag{3.1}$$

The form of the substitution (2.9) permits no change in the boundary conditions (2.5) at $\theta = 0$, if $P_0(\xi)$ and $Q_0(\xi)$ are to be independent of λ. Equation (2.10) becomes

$$P_0''' = 0, \tag{3.2}$$

and hence

$$P_0(\xi) = \tfrac{1}{2}A\xi^2, \tag{3.3}$$

where A is a constant. For small θ, $p_0'' \simeq B''(0)$ and the solutions will match as required if

$$A = B''(0) = 1{\cdot}32823. \tag{3.4}$$

Equation (2.11) now becomes

$$Q_0'' + \tfrac{1}{2}A\xi^2 Q_0' - A\xi Q_0 = 0, \tag{3.5}$$

and the solution for which $Q_0''(\infty) = A$, as required to match with q_0, is

$$Q_0(\xi) = K\xi\, {}_1F_1(-\tfrac{1}{3}, \tfrac{4}{3}, -\tfrac{1}{6}A\xi^3), \tag{3.6}$$

where ${}_1F_1$ is the confluent hypergeometric function and

$$K = \frac{1}{2}\frac{(\frac{2}{3})!}{(\frac{1}{3})!}A^{\frac{2}{3}}6^{\frac{1}{3}} = 1{\cdot}1098. \tag{3.7}$$

This shows that $Q_0'(0) = K$.

The asymptotic form of the confluent hypergeometric function for ξ large shows that

$$Q_0 \sim \tfrac{1}{2}A\xi^2 + \tfrac{2}{3}\xi^{-1} + \ldots, \tag{3.8}$$

which requires that for θ small

$$q = \tfrac{1}{2}A\theta^2 + \tfrac{2}{3}\lambda^{-1}\theta^{-1} + \ldots. \tag{3.9}$$

Likewise for θ small

$$B(\theta) = \tfrac{1}{2}A\theta^2 - \tfrac{1}{120}A^2\theta^5 + \ldots, \tag{3.10}$$

and hence for ξ large

$$P, Q \sim \tfrac{1}{2}A\xi^2 - \tfrac{1}{120}A^2\lambda^{-1}\xi^5 + \ldots. \tag{3.11}$$

These expressions indicate the nature of the extra terms which must be added to the inner and outer solutions in order to improve the matching. In fact rather more is needed. The appropriate expansions are

$$\left.\begin{aligned} p(\theta) &= p_0(\theta) + \lambda^{-1}\log\lambda\, p_1(\theta) + \lambda^{-1}p_2(\theta) + \ldots, \\ P(\xi) &= P_0(\xi) + \lambda^{-1}\log\lambda\, P_1(\xi) + \lambda^{-1}P_2(\xi) + \ldots, \end{aligned}\right\} \tag{3.12}$$

and similarly for $q(\theta)$ and $Q(\xi)$. The functions occurring in these expansions are independent of λ but do involve the parameter β. The necessity for including terms in $\lambda^{-1}\log\lambda$ is seen from a study of the appropriate equations, which are obtained by equating to zero the coefficients of successive powers of λ in (2.3), (2.4), (2.10) and (2.11). The equations are as follows:

$$(1-\beta)p_1'''' + p_0p_1'' + p_0''p_1 - \beta q_0q_1'' - \beta q_0''q_1 = 0, \tag{3.13}$$

$$p_0q_1' - p_0'q_1 + p_1q_0' - p_1'q_0 = 0, \tag{3.14}$$

$$P_1'''' = 0, \tag{3.15}$$

$$Q_1'' + P_0Q_1' - P_0'Q_1 + P_1Q_0' - P_1'Q_0 = 0, \tag{3.16}$$

$$(1-\beta)p_2'''' + p_0p_2'' + p_0''p_2 - \beta q_0q_2'' - \beta q_0''q_2 = 0, \tag{3.17}$$

$$q_0'' + p_0q_2' - p_0'q_2 + p_2q_0' - p_2'q_0 = 0, \tag{3.18}$$

$$(1-\beta)P_2'''' + P_0P_0'' - \beta Q_0Q_0'' = 0, \tag{3.19}$$

$$Q_2'' + P_0Q_2' - P_0'Q_2 + P_2Q_0' - P_2'Q_0 = 0. \tag{3.20}$$

Let us examine the behaviour of (3.19) for large ξ. From (3.3) and (3.8),

$$P_0P_0'' - \beta Q_0Q_0'' \sim \tfrac{1}{2}A^2(1-\beta)\xi^2 - \tfrac{4}{3}A\beta\xi^{-1}$$

and hence

$$P_2'' \sim -\tfrac{1}{6}A^2\xi^3 + \tfrac{4}{3}A\frac{\beta}{1-\beta}\log\xi + C_2, \tag{3.21}$$

where C_2 is a constant. The first term is precisely as demanded by (3.11). A similar inspection of (3.20) reveals that Q_2 must also contain a logarithmic term. Use of (3.3), (3.8) and (3.21) shows that for large ξ

$$Q_2'' \sim -\tfrac{1}{6}A^2\xi^3 + \left(\frac{4}{3}\frac{\beta}{1-\beta} + \frac{8}{15}\right)A\log\xi + D_2. \tag{3.22}$$

Without the presence of the term in $\log\xi$ the coefficient of ξ^3 in (3.20) cannot be zero. Studies of (3.17) and (3.18) show in the same way that for θ small

$$p_2'' \simeq \tfrac{4}{3}A\frac{\beta}{1-\beta}\log\theta + c_2, \tag{3.23}$$

$$q_2'' \simeq \tfrac{4}{3}\theta^{-3} + \left(\frac{4}{3}\frac{\beta}{1-\beta} + \frac{8}{15}\right)A\log\theta + d_2. \tag{3.24}$$

The first term of (3.24) is in agreement with (3.9). The implication of (3.23) is that for ξ large $P''(\xi)$ must have a contribution

$$\left\{\tfrac{4}{3}A\frac{\beta}{1-\beta}(\log\xi - \tfrac{1}{3}\log\lambda) + c_2\right\}\lambda^{-1} = -\tfrac{4}{9}A\frac{\beta}{1-\beta}\lambda^{-1}\log\lambda + \left(\tfrac{4}{3}A\frac{\beta}{1-\beta}\log\xi + c_2\right)\lambda^{-1}. \tag{3.25}$$

The first term demonstrates that the expansion must include terms in $\lambda^{-1}\log\lambda$ if the inner and outer solutions are to balance. The last term is satisfied by (3.21) provided that $C_2 = c_2$. Similarly, we deduce from (3.24) that $D_2 = d_2$, and that $Q''(\xi)$ must have a contribution

$$-\left(\frac{4}{9}\frac{\beta}{1-\beta} + \frac{8}{45}\right)A\lambda^{-1}\log\lambda. \tag{3.26}$$

We now turn to equations (3.13) to (3.16). The only solution of (3.13) and (3.14) satisfying the boundary conditions is $p_1 = q_1 = 0$. The solution of (3.15) which is in agreement with (3.25) is

$$P_1 = -\tfrac{2}{9}A\frac{\beta}{1-\beta}\xi^2. \tag{3.27}$$

Equation (3.16) may then be written as

$$Q_1'' + \tfrac{1}{2}A\xi^2 Q_1' - A\xi Q_1 = \frac{4}{9}\frac{\beta}{1-\beta}Q_0'', \tag{3.28}$$

using (3.5). A particular integral of (3.28) is

$$Q_1 = \frac{4}{27}\frac{\beta}{1-\beta}\xi Q_0',$$

and a complementary function is $Q_1 = Q_0$. Both of these satisfy the boundary condition $Q_1(0) = 0$, and in view of (3.26) the required solution is

$$Q_1 = \frac{4}{27}\frac{\beta}{1-\beta}(\xi Q_0' - 5Q_0) - \tfrac{8}{45}Q_0, \tag{3.29}$$

so that

$$Q_1'(0) = -\left(\frac{16}{27}\frac{\beta}{1-\beta} + \frac{8}{45}\right)K. \tag{3.30}$$

To complete this stage of the approximation the equations (3.17) to (3.20) were integrated numerically. Care was needed owing to the logarithmic behaviour of the functions, but the expressions (3.21) to (3.24) were of great assistance, and the work was shortened by the fact that all the equations have simple complementary functions. The integration of (3.17) and (3.18) under the boundary conditions

$$p_2(0) = p_2'(0) = q_2(0) = p_2'(\infty) = q_2'(\infty) = 0$$

determined the constants c_2 and d_2, and (3.19) and (3.20) were then integrated with boundary conditions

$$P_2(0) = P_2'(0) = Q_2(0) = 0, \quad P_2''(\infty) = c_2, \quad Q_2''(\infty) = d_2.$$

The results of chief importance were

$$P_2''(0) = 0{\cdot}577\frac{\beta}{1-\beta}A, \tag{3.31}$$

$$Q_2'(0) = \left(0{\cdot}557\frac{\beta}{1-\beta} - 0{\cdot}026\right)K. \tag{3.32}$$

Finally we may inquire what is the dependence on λ of the next most important terms in the expansion (3.12). It is clear that contributions in $\lambda^{-2}\log^2\lambda$ are present in equations (2.10) and (2.11), so the next terms must be of at least this order of magnitude. An inspection of the equations which govern these terms reveals no logarithmic or other awkward behaviour, so we conclude that the next terms in (3.12) are indeed multiples of $\lambda^{-2}\log^2\lambda$.

The most important physical quantities to be estimated from our analysis are the skin friction τ_W at the plate and the surface value H_t of the tangential component of magnetic intensity. The results are

$$\begin{aligned}\tau_W &= \nu\rho\left(\frac{\partial u}{\partial y}\right)_{y=0} = \tfrac{1}{4}\rho\left(\frac{U_0^3\nu}{x}\right)^{\frac{1}{2}} f''(0) = \tfrac{1}{4}\rho\left(\frac{U_0^3\nu}{x}\right)^{\frac{1}{2}}(1-\beta)^{\frac{3}{4}}P''(0)\\ &= 0{\cdot}33206\rho\left(\frac{U_0^3\nu}{x}\right)^{\frac{1}{2}}(1-\beta)^{\frac{3}{4}}\left\{1-0{\cdot}4444\frac{\beta}{1-\beta}\lambda^{-1}\log\lambda\right.\\ &\qquad\left.+0{\cdot}577\frac{\beta}{1-\beta}\lambda^{-1}+O(\lambda^{-2}\log^2\lambda)\right\}\\ &= 0{\cdot}33206\rho\left(\frac{U_0^3\nu}{x}\right)^{\frac{1}{2}}(1-\beta)^{\frac{1}{2}}\left\{1-0{\cdot}4444\beta\epsilon^{-1}\log\frac{\epsilon}{1-\beta}+0{\cdot}577\beta\epsilon^{-1}+O(\epsilon^{-2}\log^2\epsilon)\right\},\end{aligned} \tag{3.33}$$

$$\begin{aligned}H_t &= \tfrac{1}{2}H_0 g'(0) = \tfrac{1}{2}H_0\lambda^{-\frac{1}{4}}Q'(0)\\ &= 0{\cdot}5549H_0\lambda^{-\frac{1}{4}}\left\{1-\left(0{\cdot}5926\frac{\beta}{1-\beta}+0{\cdot}1778\right)\lambda^{-1}\log\lambda\right.\\ &\qquad\left.+\left(0{\cdot}557\frac{\beta}{1-\beta}-0{\cdot}026\right)\lambda^{-1}+O(\lambda^{-2}\log^2\lambda)\right\}\\ &= 0{\cdot}5549H_0(1-\beta)^{\frac{1}{4}}\epsilon^{-\frac{1}{4}}\left\{1-(0{\cdot}4148\beta+0{\cdot}1778)\epsilon^{-1}\log\frac{\epsilon}{1-\beta}\right.\\ &\qquad\left.+(0{\cdot}583\beta-0{\cdot}026)\epsilon^{-1}+O(\epsilon^{-2}\log^2\epsilon)\right\}.\end{aligned} \tag{3.34}$$

4. Equations for small conductivity

When the electrical conductivity is such that ϵ is small compared with unity a completely fresh start is necessary, though it will be observed that the steps in the analysis have strong points of resemblance to what has gone before. The preliminary transformation of variables given by (2.1) and (2.2) is no longer required, and we revert to the original forms of equations (1.1) and (1.2).

For a non-conducting fluid in which ϵ is zero the solution of (1.2) satisfying the boundary conditions (1.3) is

$$g(\eta) = 2\eta, \tag{4.1}$$

and (1.1) becomes $f'''+ff''=0$. As before the required solution is given by the Blasius function

$$f(\eta) = B(\eta). \tag{4.2}$$

When ϵ is small but non-zero (1.2) is not satisfied by (4.1) and (4.2) for large values of η. The asymptotic form of $B(\eta)$ shows that

$$f \sim 2\eta - c \tag{4.3}$$

for η large, where $c = 1{\cdot}7208$. Consequently the second term of (1.2) has the limiting value $-2c\epsilon$, while (1.3) insists that $g''(\infty) = 0$. It is clear that we must introduce a new co-ordinate to discuss the outer part of the boundary layer. Throughout this outer layer f' and g' remain of order unity, in view of (1.3), but the terms of (1.2) must be of comparable magnitude for arbitrarily small ϵ. Accordingly we write

$$\eta = \epsilon^{-\frac{1}{2}}\zeta, \quad f(\eta) = \epsilon^{-\frac{1}{2}}F(\zeta), \quad g(\eta) = \epsilon^{-\frac{1}{2}}G(\zeta). \tag{4.4}$$

Then $f'(\eta) = F'(\zeta), f''(\eta) = \epsilon^{\frac{1}{2}}F''(\zeta)$, etc., and (1.1) and (1.2) become

$$\epsilon F''' + FF'' - \beta GG'' = 0, \tag{4.5}$$

$$G'' + FG' - F'G = 0, \tag{4.6}$$

with boundary conditions $F'(\infty) = G'(\infty) = 2$. When ζ is small the values of F and G must be in accord with the values of f and g for η large, matching being carried out as for the case of large conductivity. The only difference is that the form of transformation given above shows that now it is the first derivatives of the corresponding inner and outer functions which must balance at each stage of the series expansion.

5. Solutions for small conductivity

The first approximations $f_0(\eta)$, $g_0(\eta)$, $F_0(\zeta)$ and $G_0(\zeta)$ are found by taking ϵ to be zero in (1.1), (1.2), (4.5) and (4.6). The required solutions of (4.5) and (4.6) are seen at once to be

$$F_0 = G_0 = 2\zeta. \tag{5.1}$$

Matching these solutions requires no change in the boundary conditions (1.3) at $\eta = \infty$, so, as in (4.1) and (4.2),

$$f_0 = B(\eta), \quad g_0 = 2\eta. \tag{5.2}$$

The only one of these four functions which gives a contribution to later stages of the matching is f_0. As in (4.3), $f_0 \sim 2\eta - c$ and this implies that

$$F = 2\zeta - c\epsilon^{\frac{1}{2}} + \dots \tag{5.3}$$

for ζ small.

Appropriate forms of expansion turn out to be

$$\left.\begin{aligned} f(\eta) &= f_0(\eta) + \epsilon^{\frac{1}{2}} f_1(\eta) + \epsilon \log \epsilon f_2(\eta) + \epsilon f_3(\eta) + \dots, \\ F(\zeta) &= F_0(\zeta) + \epsilon^{\frac{1}{2}} F_1(\zeta) + \epsilon \log \epsilon F_2(\zeta) + \epsilon F_3(\zeta) + \dots, \end{aligned}\right\} \tag{5.4}$$

and similarly for $g(\eta)$ and $G(\zeta)$. The presence of the logarithmic terms will be proved in due course. The terms in $\epsilon^{\frac{1}{2}}$ in (1.1), (1.2), (4.5) and (4.6) lead to the equations

$$f_1''' + f_0 f_1'' + f_0'' f_1 - \beta g_0 g_1'' - \beta g_0'' g_1 = 0, \tag{5.5}$$

$$g_1'' = 0, \tag{5.6}$$

$$F_0 F_1'' + F_0'' F_1 - \beta G_0 G_1'' - \beta G_0'' G_1 = 0, \tag{5.7}$$

$$G_1'' + F_0 G_1' - F_0' G_1 + F_1 G_0' - F_1' G_0 = 0. \tag{5.8}$$

Using (5.1) we can write (5.7) as

$$2\zeta(F_1'' - \beta G_1'') = 0. \tag{5.9}$$

Now $F_1'(\infty) = G_1'(\infty) = 0$, and by (5.3), $F_1(0) = -c$, $G_1(0) = 0$, so (5.9) may be integrated twice to give

$$F_1 = \beta G_1 - c. \tag{5.10}$$

Substituting this result in (5.8) we obtain

$$G_1'' + 2(1-\beta)\zeta G_1' - 2(1-\beta) G_1 = 2c. \tag{5.11}$$

A particular integral of this equation is $G_1 = -c/(1-\beta)$, and the complementary functions are

$$G_1 = \zeta \quad \text{and} \quad G_1 = \exp[-(1-\beta)\zeta^2] - 2(1-\beta)^{\frac{1}{2}}\zeta\,\text{erfc}\,(1-\beta)^{\frac{1}{2}}\zeta,$$

where
$$\text{erfc}\,x = \int_x^\infty e^{-u^2}\,du.$$

The boundary conditions show that the required solution of (5.11) is

$$G_1 = \frac{c}{1-\beta}\{-1+\exp[-(1-\beta)\zeta^2] - 2(1-\beta)^{\frac{1}{2}}\zeta\,\text{erfc}\,(1-\beta)^{\frac{1}{2}}\zeta\}, \tag{5.12}$$

and hence

$$F_1 = \frac{c}{1-\beta}\{-1+\beta\exp[-(1-\beta)\zeta^2] - 2\beta(1-\beta)^{\frac{1}{2}}\zeta\,\text{erfc}\,(1-\beta)^{\frac{1}{2}}\zeta\}. \tag{5.13}$$

Then
$$F_1' = \beta G_1' = -\frac{2c\beta}{(1-\beta)^{\frac{1}{2}}}\,\text{erfc}\,(1-\beta)^{\frac{1}{2}}\zeta, \tag{5.14}$$

and in particular

$$F_1'(0) = \beta G_1'(0) = -\frac{\pi^{\frac{1}{2}}c\beta}{(1-\beta)^{\frac{1}{2}}} \tag{5.15}$$

since $\text{erfc}\,0 = \frac{1}{2}\pi^{\frac{1}{2}}$. The solution of (5.6) to give the necessary matching is

$$g_1 = -\frac{\pi^{\frac{1}{2}}c\eta}{(1-\beta)^{\frac{1}{2}}}. \tag{5.16}$$

Equation (5.5) reduces to $f_1''' + Bf_1'' + B''f_1 = 0$, and the required solution is

$$f_1 = -\frac{\pi^{\frac{1}{2}}c\beta}{4(1-\beta)^{\frac{1}{2}}}(B+\eta B'). \tag{5.17}$$

We note that

$$g_1'(0) = -\frac{\pi^{\frac{1}{2}}c}{(1-\beta)^{\frac{1}{2}}}, \quad f_1''(0) = -\frac{3}{4}\frac{\pi^{\frac{1}{2}}c\beta}{(1-\beta)^{\frac{1}{2}}}A, \tag{5.18}$$

where A is given by (3.4). The contributions to later terms of the series implied by these results are that for ζ small F has a term $\frac{1}{4}\pi^{\frac{1}{2}}c^2\beta(1-\beta)^{-\frac{1}{2}}\epsilon$, and for η large f has a term $c\beta\eta^2\epsilon$ and g a term $c\eta^2\epsilon$. The last two results follow from the fact that $F_1''(0) = \beta G_1''(0) = 2c\beta$.

The equations which determine the next batch of functions are:

$$f_2''' + f_0 f_2'' + f_0'' f_2 - \beta g_0 g_2'' - \beta g_0'' g_2 = 0, \tag{5.19}$$

$$g_2'' = 0, \tag{5.20}$$

$$F_0 F_2'' + F_0'' F_2 - \beta G_0 G_2'' - \beta G_0'' G_2 = 0, \tag{5.21}$$

$$G_2'' + F_0 G_2' - F_0' G_2 + F_2 G_0' - F_2' G_0 = 0, \tag{5.22}$$

$$f_3''' + f_0 f_3'' + f_0'' f_3 - \beta g_0 g_3'' - \beta g_0'' g_3 + f_1 f_1'' - \beta g_1 g_1'' = 0, \tag{5.23}$$

$$g_3'' + f_0 g_0' - f_0' g_0 = 0, \tag{5.24}$$

$$F_0''' + F_0 F_3'' + F_0'' F_3 - \beta G_0 G_3'' - \beta G_0'' G_3 + F_1 F_1'' - \beta G_1 G_1'' = 0, \tag{5.25}$$

$$G_3'' + F_0 G_3' - F_0' G_3 + F_3 G_0' - F_3' G_0 + F_1 G_1' - F_1' G_1 = 0. \tag{5.26}$$

Equation (5.24) is $g_3'' + 2B - 2\eta B' = 0$, which shows that when η is large $g_3 \sim c\eta^2$, as predicted above. For large η, (5.23) now gives $f_3'''' + (2\eta - c) f_3''' - 4c\beta\eta \sim 0$ which can only be satisfied if $f_3''' \sim 2c\beta + c^2\beta\eta^{-1}$ and hence

$$f_3' \sim 2c\beta\eta + c^2\beta\log\eta. \tag{5.27}$$

The first term is as required for the matching and the second proves (in the same way as for large ϵ) that logarithmic terms must be present in the expansions. Similarly, a study of (5.25) and (5.26) shows that when ζ is small

$$F_3' \simeq c^2\beta\log\zeta, \tag{5.28}$$

but G_3' is bounded. Now $\log\zeta = \log\eta + \frac{1}{2}\log\epsilon$, and so (5.28) implies that an additional contribution

$$f'(\eta) \simeq \tfrac{1}{2}c^2\beta\epsilon\log\epsilon \tag{5.29}$$

for η large is required to balance the inner and outer solutions.

The only solutions of (5.21) and (5.22) which satisfy the boundary conditions are $F_2 = G_2 = 0$. Also since g_3 and G_3 have no logarithmic terms the required solution of (5.20) is $g_2 = 0$. Equation (5.19) reduces to $f_2''' + Bf_2'' + B''f_2 = 0$, and in view of (5.29) the appropriate solution is

$$f_2 = \tfrac{1}{8}c^2\beta(B + \eta B'), \tag{5.30}$$

which shows that

$$f_2''(0) = \tfrac{3}{8}c^2\beta A. \tag{5.31}$$

In the set of equations corresponding to the next stage of the approximation (5.25) and (5.26) can be integrated explicitly, using the known boundary conditions

$$F_3(0) = \tfrac{1}{4}\pi^{\frac{1}{2}}c^2\beta(1-\beta)^{-\frac{1}{2}}, \quad G_3(0) = F_3'(\infty) = G_3'(\infty) = 0.$$

The values of $F_3'(0)$ and $G_3'(0)$ thus obtained, together with the conditions $f_3(0) = f_3'(0) = g_3(0) = 0$, suffice to determine the required solutions of (5.23) and (5.24), but here, as at the corresponding stage of the work for large conductivity, numerical integration is needed for the first time. As before, simple complementary functions are available. The integrations gave the values

$$f_3''(0) = \left[11{\cdot}814 + 0{\cdot}606\frac{\beta}{1-\beta} + \tfrac{3}{8}c^2\log(1-\beta)\right]\beta A, \tag{5.32}$$

$$g_3'(0) = 6{\cdot}056 - 0{\cdot}354\frac{\beta}{1-\beta}. \tag{5.33}$$

Further terms in the expansions could be calculated if required, but it seems unlikely that the range of values of ϵ for which the series solution gives useful information could be extended significantly without considerable labour. An inspection of the equations governing the next terms of the expansion shows that these are not of greater order of magnitude than $\epsilon^{\frac{3}{2}}\log\epsilon$.

The formulae for τ_W and H_t which follow from the analysis are

$$\begin{aligned}\tau_W &= \tfrac{1}{4}\rho\left(\frac{U_0^3\nu}{x}\right)^{\frac{1}{2}} f''(0)\\ &= 0{\cdot}33206\rho\left(\frac{U_0^3\nu}{x}\right)^{\frac{1}{2}}\left\{1 - 2{\cdot}2875\frac{\beta}{(1-\beta)^{\frac{1}{2}}}\epsilon^{\frac{1}{2}} + 1{\cdot}1104\beta\epsilon\log(1-\beta)\epsilon\right.\\ &\qquad\left. + 11{\cdot}814\beta\epsilon + 0{\cdot}606\frac{\beta^2}{1-\beta}\epsilon + O(\epsilon^{\frac{3}{2}}\log\epsilon)\right\},\end{aligned} \tag{5.34}$$

$$\begin{aligned}H_t &= \tfrac{1}{2}H_0 g'(0)\\ &= H_0\left\{1 - 1{\cdot}5250(1-\beta)^{-\frac{1}{2}}\epsilon^{\frac{1}{2}} + 3{\cdot}028\epsilon - 0{\cdot}177\frac{\beta}{1-\beta}\epsilon + O(\epsilon^{\frac{3}{2}}\log\epsilon)\right\}.\end{aligned} \tag{5.35}$$

6. Discussion of results

The chief results of the analysis are contained in equations (3.33), (3.34), (5.34) and (5.35). These give the skin-friction τ_W and the surface value H_t of the tangential component of magnetic intensity as functions of the parameters ϵ and β. As shown in the Appendix the current flow in the boundary layer per unit length of the plate is $H_t - H_0$, so this important quantity is also determined by H_t.

Probably the main interest centres on the forms of the leading terms of each expansion, since these indicate to what extent theories which assume the conductivity to be either infinite or zero are likely to be reliable. The perfect conductivity solutions (2.6) and (2.8) give the correct limiting skin-friction as from (3.33), and give $H_t = 0$. Since the leading term of (3.34) is proportional only to $\epsilon^{-\frac{1}{2}}$ this last result may be inadequate except at very large conductivities. For zero conductivity the solutions (4.1) and (4.2) give correctly the leading terms of (5.34) and (5.35). In each case the first correction term is proportional to $\epsilon^{\frac{1}{2}}$.

For large conductivity the values given by (3.33) and (3.34) should be adequate for all practical purposes (say within 1 %) for $\epsilon > 10$, and should provide useful estimates (say within 10 %) for $\epsilon > 3$. For $\epsilon = 10$, (3.33) gives values of τ_W which are indistinguishable from those shown in figure 2 of **I**, which were obtained by direct numerical integration. The series for small conductivity (5.34) and (5.35) are less efficient. The ratio of the magnitudes of successive groups of terms is $\epsilon^{\frac{1}{2}}$ (ignoring logarithmic factors) compared with ϵ^{-1} for large conductivity, and in addition the numerical coefficients increase rapidly. The series should, however, be entirely adequate for $\epsilon < 0{\cdot}001$ and useful for $\epsilon < 0{\cdot}01$. A comparison with the computed values of τ_W in figure 2 of **I** for $\epsilon = 0{\cdot}005$ shows detectable but small discrepancies.

The formulae (3.33), (3.34), (5.34) and (5.35) each have a singular behaviour at $\beta = 1$, but it would be rash to draw too precise conclusions since in the analysis $1-\beta$ was assumed not to be small. A direct study of the original equations is informative. For any value of ϵ it follows from (1.2) and (1.3) that when η is large $f \sim g \sim 2(\eta - a)$, where a is some constant, and from (1.1) and (1.2) it is now readily shown that

$$f'' \sim b\, e^{-c(\eta-a)^2}, \quad g'' \sim d\, e^{-c(\eta-a)^2}, \tag{6.1}$$

where b and d are constants and c is the algebraically smaller root of the equation

$$c^2 - (1+\epsilon)\, c + (1-\beta)\, \epsilon = 0. \tag{6.2}$$

Both roots are positive for $\beta > 1$, and when $1-\beta$ is small the smaller root is approximately $c = (1-\beta)\,\epsilon/(1+\epsilon)$. This shows that the width of the boundary layer is given by $(1-\beta)\,(\eta-a)^2 = O(1)$, or

$$\frac{y}{x} = O[(1-\beta)^{-\frac{1}{2}} R^{-\frac{1}{2}}], \tag{6.3}$$

where R is the Reynolds number $U_0 x/\nu$. The boundary-layer equations are valid only if y/x is small within the boundary layer, and consequently they cease to hold when $1-\beta = O(R^{-1})$. Incidentally the asymptotic forms given above show clearly that solutions of the present type cannot exist for $\beta > 1$. Equation (6.2) would then imply that for η large, f'' and g'' increase exponentially with η, which is impossible.

A point which troubled the authors of **I** was that the Oseen analysis predicted that for large finite ϵ the value of τ_W exceeds that for $\epsilon = \infty$, at the same value of β, while their numerical integrations of the full equations showed that in fact τ_W is reduced. The leading correction term in (3.33), that multiplied by $\epsilon^{-1}\log\epsilon$, clearly acts to reduce τ_W. Actually an inspection of the results of the Oseen analysis given in **I** shows that they are very inaccurate for all ϵ, large and small. However, the authors state that their numerical values could be improved by supposing that the convective velocity and magnetic fields, instead of being equal to their free-stream values, are suitably chosen fractions of them. It is not evident that a major improvement over the whole range of the parameters can be achieved in this way, particularly in view of the involved forms of the solutions found above. Failing such an improvement, it seems that the linearized method of analysis is not to be relied on in a problem of this complexity.

Appendix. Derivation of the boundary-layer equations

The equations governing steady incompressible magnetohydrodynamic flow are

$$(\mathbf{q}\,.\nabla)\,\mathbf{q} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{q} + \frac{\mu}{\rho}\mathbf{j}\wedge\mathbf{H}, \tag{A 1}$$

$$\frac{1}{\sigma}\mathbf{j} = \mathbf{E} + \mu\mathbf{q}\wedge\mathbf{H}, \tag{A 2}$$

$$\mathbf{j} = \operatorname{curl}\mathbf{H}, \tag{A 3}$$

$$\operatorname{div}\mathbf{q} = \operatorname{div}\mathbf{H} = \operatorname{div}\mathbf{E} = \operatorname{curl}\mathbf{E} = 0, \tag{A 4}$$

in m.k.s. units, where p is the pressure, $\mathbf{q}$ the velocity, $\mathbf{j}$ the current, and $\mathbf{H}$ and $\mathbf{E}$ the magnetic and electric intensities. The boundary conditions in our problem are that $\mathbf{q} = 0$ and $H_y = 0$ on the plate $y = 0$, $x > 0$, and at a large distance $p = p_0$, $\mathbf{q} = U_0\mathbf{i}$, $\mathbf{H} = H_0\mathbf{i}$. (Some comments on the plate condition $H_y = 0$ are made below.) The electric field $\mathbf{E}$ may be taken to be zero everywhere, and using (A 3) equations (A 1) and (A 2) may be written as

$$(\mathbf{q}\,.\nabla)\,\mathbf{q} = -\frac{1}{\rho}\nabla(p + \tfrac{1}{2}\mu\mathbf{H}^2) + \nu\nabla^2\mathbf{q} + \frac{\mu}{\rho}(\mathbf{H}\,.\nabla)\,\mathbf{H}, \tag{A 5}$$

$$\mathbf{q}\wedge\mathbf{H} = \frac{1}{\sigma\mu}\operatorname{curl}\mathbf{H}. \tag{A 6}$$

We require two-dimensional solutions and in view of (A 4) we may look for these in the forms

$$\mathbf{q} = \operatorname{curl}\psi(x,y)\,\mathbf{k}, \quad \mathbf{H} = \operatorname{curl}A(x,y)\,\mathbf{k}, \tag{A 7}$$

where $\mathbf{k}$ is a unit vector in the z-direction.

On the usual boundary-layer assumption that a rate of change in the y-direction is of a greater order of magnitude than one in the x-direction, the y-component of (A 5) shows that $p + \frac{1}{2}\mu\mathbf{H}^2$ has negligible variation across the boundary layer, and so may be taken to have the constant value $p_0 + \frac{1}{2}\mu H_0^2$ everywhere.

The boundary-layer approximations to the x-component of (A 5) and to (A 6) then give

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\nu\frac{\partial^3\psi}{\partial y^3}+\frac{\mu}{\rho}\left(\frac{\partial A}{\partial y}\frac{\partial^2 A}{\partial x\partial y}-\frac{\partial A}{\partial x}\frac{\partial^2 A}{\partial y^2}\right), \tag{A 8}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial A}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial A}{\partial y}=\frac{1}{\sigma\mu}\frac{\partial^2 A}{\partial y^2}. \tag{A 9}$$

Guided by the form of the Blasius solution for a non-conducting fluid we next write

$$\psi=(U_0\nu x)^{\frac{1}{2}}f(\eta), \quad A=H_0(\nu x/U_0)^{\frac{1}{2}}g(\eta), \tag{A 10}$$

where $\eta=\frac{1}{2}(U_0/\nu x)^{\frac{1}{2}}y$. Equations (A 8) and (A 9) become ordinary differential equations in η, with the forms given in (1.1) and (1.2). We see also from (A 7) and (A 10) that the velocity and magnetic field components are as given in (1.5) and (1.6), and that the boundary conditions stated above imply the conditions (1.3). The boundary-layer approximation to (A 3) is that $\mathbf{j}=j\mathbf{k}$, where

$$j=-\frac{\partial^2 A}{\partial y^2}=-\tfrac{1}{4}H_0\left(\frac{U_0}{\nu x}\right)^{\frac{1}{2}}g''(\eta), \tag{A 11}$$

and the total current in the boundary layer, flowing parallel to the edge of the plate, is

$$\int_0^\infty j\,dy=\left[-\frac{\partial A}{\partial y}\right]_0^\infty=H_l-H_0 \tag{A 12}$$

per unit distance in the x-direction, where H_l is the value of H_x at the plate $y=0$.

The boundary condition $H_y=0$ on $y=0$, $x>0$ is obviously essential for an infinitely thin plate with a similar boundary layer on its lower side. By symmetry the values of H_y at $y=0$ on the upper and lower surfaces are equal and opposite, and the necessary continuity of μH_y across the plate surfaces can be achieved only if $H_y=0$ at the plate. But in fact the resulting form of solution is of wider applicability. Equation (1.6) shows that when $y=0$, $H_x=H_l=\frac{1}{2}H_0g'(0)$ which is independent of x. Consequently for a plate of arbitrary thickness all the magnetic equations are satisfied if $\mathbf{H}$ has the constant value $H_l\mathbf{i}$ throughout the plate.

Finally it may be noted that for the boundary-layer approximation to be appropriate it is not sufficient that the Reynolds number $R=U_0x/\nu$ shall be large. By comparing the forms of (A 9) and (A 8), or by studying the width of the outer layer for small conductivity as found in § 5, we see that it is also necessary that the magnetic Reynolds number $R_M=\epsilon R=U_0x\sigma\mu$ shall be large compared with unity. This sets a lower limit to the conductivities for which the results of this paper are applicable.

REFERENCES

CARRIER, G. F. & GREENSPAN, H. P. 1960 *J. Fluid Mech.* **7**, 22.

GREENSPAN, H. P. 1960 *Phys. Fluids* **3**, 581.

GREENSPAN, H. P. & CARRIER, G. F. 1959 *J. Fluid Mech.* **6**, 77 (**I**).

MORGAN, G. W., PIPKIN, A. C. & WARNER, W. H. 1958 *J. Aero. Sci.* **25**, 173.

MORGAN, G. W. & WARNER, W. H. 1956 *J. Aero. Sci.* **23**, 937.